We shall omit here and in general in what follows, all theorems obtained by dualization.

**Theorem II.** *The lines joining pairs of vertices of two quadrangles with the same diagonal triangle form a quadrilateral with this diagonal triangle.*

Let us consider Fig. 1 again. $QR$ was chosen originally as an arbitrary ray through $Q$. Hence, the demonstration of Theorem I includes the proof that, if $QK$ and $QK'$ be arbitrary rays through $Q$, $KK'$ and $LL'$ will meet on $q$, the polar of $Q$. So, also, will $K'L$ and $KL'$. Hence, in the quadrilateral formed by $KK'$, $LL'$, $MM'$, and $NN'$, we have $KK'$ and $LL'$ intersecting on $q$ and similarly $MM'$ and $NN'$. Likewise, we may show that $KK'$ and $NN'$, and $LL'$ and $MM'$ intersect on $s$, and finally $KK'$ and $MM'$, and $LL'$ and $NN'$ on $r$. Hence, the diagonal triangle of the quadrilateral is $QRS$, the same as that of either quadrangle $KLMN$ or $K'L'M'N'$.

If now $K'$ approaches $K$ along the conic, $L'$, $M'$, and $N'$ approach $L$, $M$, and $N$ respectively and the sides of the quadrilateral approach as limiting positions, the tangents to the conic at the vertices of the quadrangle. From which follows

**Theorem III.** *The tangents at any four points of a conic form a quadrilateral which has the same diagonal triangle as the quadrangle of those four points.*

By combining this theorem with the dual of Theorem I, viz., the eight sides of two quadrilaterals with the same diagonal triangle envelop a conic, we obtain the following

**Theorem IV.** *The eight tangents at the points of intersection of two conics have the same diagonal triangle as the quadrilateral of the four tangents to either one of the conics, at these points. These two quadrilaterals have the same diagonal triangle and, consequently, their sides envelop a conic.*

The second part of the theorem is the dual of the first. The conic mentioned in the second part is the covariant conic[1] of the two given conics, and is the locus of points from which the pairs of tangents to the given conics are harmonic conjugates of each other. A synthetic discussion has been given by Professor Lehmer.[2]

**Theorem V.** *The quadrilateral of the four common tangents to two conics has the same diagonal triangle as the quadrangle of the intersection points of the two conics.*

For let $PP'$ be one of these common tangents, $P$ and $P'$ being the respective points of contact. Then if $Q$, $R$, $S$; $Q'$, $R'$, $S'$, respectively, be the remaining vertices of the quadrangles determined by $P$ and $P'$, these quadrangles having the same diagonal triangle as the quadrangle of the four points of intersection of the two conics, then $QQ'$, $RR'$, and $SS'$ will be the remaining sides of the quadrilateral determined by $PP'$, having the same diagonal triangle as the quadrangles just determined. $Q$, $R$, $S$ will lie on the first conic; $Q'$, $R'$, $S'$ on the second; and $QQ'$, $RR'$, $SS'$ will be tangent to both.

[1] See Salmon, *Conic Sections*, pp. 306 and 344.

[2] *Amer. Math. Monthly*, Feb., 1908, "A Discussion by Synthetic Methods of the Covariant Conic of two Given Conics."

## Section 2. Involutory Quadratic Transformation. Conics Through Four Given Points and Tangent to a Given Line.

An involution of rays may be regarded as composed of two superposed projective pencils of rays where the correspondence is such that a given ray has but one corresponding ray whether it be considered as belonging to one or the other of the superposed pencils. Consider two such involutions of rays with centers at $A$ and $B$ (Fig. 3), to be called the involutions $A$ and $B$. To a point $Q$ of the plane corresponds a point $Q'$ obtained by finding the point of intersection of $AQ'$ and $BQ'$, the rays at $A$ and $B$ corresponding to $AQ$ and $BQ$ respectively. If a point $Q$ moves along a ray $q$, it projects to $A$ and $B$ in two perspective pencils

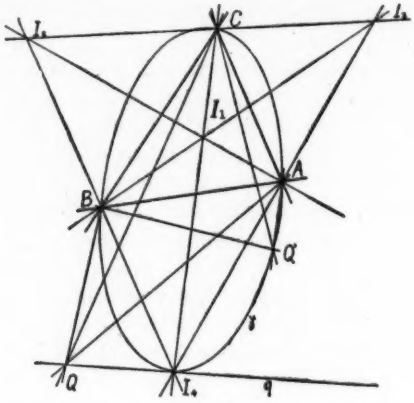

Fig. 3.

of rays. The corresponding rays $AQ'$ and $BQ'$ in the involutions $A$ and $B$ describe pencils of rays projective to the pencils $AQ$ and $BQ$ respectively and hence projective to each other. $Q'$, therefore, describes a conic through $A$ and $B$. It has been noted that to every point $Q$ of the plane there is a corresponding point $Q'$, and conversely. There are, however, special points and lines for which the correspondence is not unique. Any point of the line $AB$, for instance, corresponds to the same point $C$, the point of intersection of the rays at $A$ and $B$ corresponding to $AB$ considered first as a ray of the involution $A$ and then as one of the involution $B$. Conversely, $AB$ may be considered as corresponding to $C$.

Denote now by $\gamma$ the conic described by $Q'$, as $Q$ describes $q$. As $Q'$ describes $\gamma$, the pencils of rays $CQ'$ and $CQ$ are projective to each other. For the pencil $CQ'$ is projective to the pencil $AQ'$ and the pencil $CQ$ is perspective to the pencil $AQ$, the pencils $AQ$ and $AQ'$ are projective to each other by hypothesis. Moreover, $Q$ and $Q'$ are related to each other reciprocally by means of the involutions $A$ and $B$. Hence, the pencils $CQ'$ and $CQ$, superposed at $C$, give an involution of rays at that point which may be paired with the involution $A$ or $B$ to give the original transformation. Suppose that $A$ and $B$, which were chosen arbitrarily,

have double rays. The intersection point of a double ray of $A$ with a double ray of $B$ is a self-corresponding point in this transformation of the plane. Hence, it must also lie on a double ray of $C$. Therefore, the involution $C$ has double rays, and two of the intersections of the double rays of $A$ and $B$ lie on one of them, while the other two lie on the other. Hence, the six double rays of $A$, $B$, and $C$ intersect in four points and, consequently, form a quadrangle whose diagonal triangle is $ABC$. Denote the four intersection points by $I_1$, $I_2$, $I_3$, $I_4$. These are obviously the only self-corresponding points of the plane. It should also be noted that since the involution $C$ has been added, all the vertices of the triangle $ABC$ correspond to its opposite sides, and conversely.[1]

Now, if with the involutions $A$ and $B$ arbitrarily chosen and the involution at $C$ determined as above, we consider the point $Q$ moving on an arbitrary ray $q$, its corresponding point describes a conic $\gamma$, through $A$, $B$, and $C$. If, as $Q$ describes $q$, $Q'$ crosses $q$, $Q$ and $Q'$ are at that instant the points of intersection of $\gamma$ and $q$. Hence the

**Theorem VI.** *The conic into which a line is converted by the involutory quadratic transformation meets this line in a pair of corresponding points, the only pair of such points on the line. If the line is tangent to its corresponding conic, the point of tangency is a self-corresponding point of the transformation.*

*Corollary.* The conic corresponding to a line through a self-corresponding point is the conic through $A$, $B$, $C$ which is tangent to the given line at the self-corresponding point. The conic of Fig. 3 is drawn in this position for convenience.

Moreover, any two corresponding rays of an involution are harmonic conjugates of the double rays. Hence, the rays joining $Q$ and $Q'$ to $A$, $B$, and $C$ are harmonic conjugates of the double rays at those points. Since the pencil of conics through the self-corresponding points $I_1$, $I_2$, $I_3$, $I_4$ cuts out an involution of points on any ray of the plane, in particular on $q$, the double points of this involution mark the points of tangency of the two conics of the pencil tangent to the line. The three pairs of opposite sides of the quadrangle of the self-corresponding points are the pairs of double rays of the three involutions $A$, $B$, and $C$; and they cut $q$ in three pairs of points of the involution. Now, since $Q$ and $Q'$ on $q$ are harmonic conjugates of each pair of these points, they are the double points sought for. Whence, we write

**Theorem VII.** *The line joining a pair of corresponding points, $Q$ and $Q'$, in an involutory quadratic tranformation of the plane, is tangent to two conics of the pencil of conics through the four self-corresponding points of the transformation, the points of tangency being $Q$ and $Q'$.*

Conversely, *if there are two conics of this pencil of conics tangent to an arbitrary line of the plane, the points of tangency are corresponding points in the transformation.*

The reality of the conics of the pencil through $I_1$, $I_2$, $I_3$, $I_4$, tangent to a given line, depends on whether the involution of points on that line determined by the

[1] For a more complete discussion of this transformation see Professor Lehmer's article in AMER. MATH. MONTHLY, "On the Combination of Involutions," March, 1911.

pencil of conics has double points or not. The intersections of the line with the opposite sides of the quadrangle give pairs of corresponding points on the line. These pairs of points must not divide each other if the involution is to have double points. Therefore, the given line must cut each pair of opposite sides of the quadrangle of the four points either in the segments upon which a vertex of the diagonal triangle lies, or else in those segments upon which it does not lie. If this condition is fulfilled for one pair of sides, it is fulfilled for the remaining pairs.

In this transformation, the points of intersection of the given line with the sides of the quadrangle of the self-corresponding points are points on the double rays of $A$, $B$, and $C$. A point on one of these double rays is evidently converted into a point on that same ray,—its harmonic conjugate with respect to the two self-corresponding points on the ray.

**Theorem VIII.** *The conic into which a given line is converted by the involutory quadratic transformation passes through the harmonic conjugates of the intersections of that line with the six sides of the quadrangle whose vertices are* $I_1$, $I_2$, $I_3$, $I_4$.

The conic corresponding to the line at infinity is then the conic passing through the middle points of the sides of the quadrangle and through the vertices of its diagonal triangle.

It is to be noted that a pair of corresponding points, $Q$ and $Q'$, together with the vertices of the diagonal triangle, completely determine the quadratic transformation, as follows: $Q$ and $Q'$ must project to $A$ (Fig. 3), for example, in a pair of corresponding rays. Likewise, the rays $AB$ and $AC$ are corresponding. This determines the involution at $A$. Similarly, at $B$ and $C$. The self-corresponding points may be found as intersections of the double rays of the several involutions.

## Section 3. Conics with Three Tangents and Two Points Given.

The proof for the number of conics with three points and two tangents given has been established by Steiner.[1] The following proof is carried out more or less along the same lines. The problem at hand is the number and construction of the conics fulfilling the given conditions, the solution of which depends on an auxiliary theorem, which will be discussed first.

**Theorem IX.** *The range of conics tangent to four lines determines at any point of the plane an involution of rays composed of the pairs of tangents from the points to the conics of the range. If the four fixed tangents should coincide in pairs so that the range of conics consists of those tangent to two fixed lines with fixed points of contact, the involution at any point of the plane has double rays.*

For, in particular, the line joining this point to the point of intersection of the two fixed lines is a double ray of the involution. This follows from the fact that the three pairs of opposite vertices of the complete quadrilateral of the four fixed lines, being the degenerate conics of the range, project to any point $S$ in three pairs of corresponding rays of the involution determined at $S$. In our case, two pairs of opposite vertices have fallen together in one point, the point of intersection of the fixed rays, which therefore projects as a double ray.

[1] *Vorlesungen über Synthetische Geometrie*, Zweiter Teil, p. 236, ed. 1876.

Apply this to the solution of the problem. Call the given lines $a$, $b$, and $c$; the opposite vertices of the triangle they determine $A$, $B$, and $C$; and the given points $L$ and $L'$ (Fig. 4). An involution of rays is determined at $A$, for instance, by the rays $b$ and $c$ as one pair and $LA$ and $L'A$ as the other. Similarly, at $B$ and $C$. Call the double rays of the involution at $A$, $g$ and $h$; of the involution at $B$, $g'$ and $h'$; of the involution at $C$, $g''$ and $h''$. Since this method of establishing the involutions determines an involutory quadratic transformation of the plane which has $L$ and $L'$ for corresponding points and has the centers of its involutions at $A$, $B$, and $C$, the intersection points of the double rays at $A$, $B$, and $C$ will be

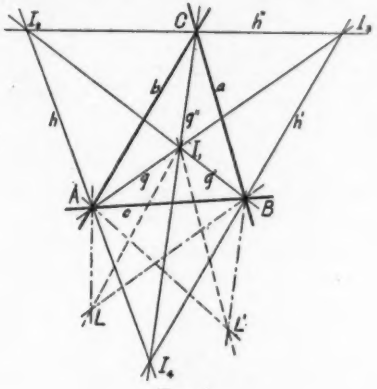


FIG. 4.

the four vertices of a complete quadrangle with diagonal triangle $ABC$. Call these intersection points $I_1$, $I_2$, $I_3$, $I_4$. They are naturally the self-corresponding points of the transformation. $I_1$ is the intersection point of double rays of $A$, $B$, and $C$; say, $g$, $g'$, and $g''$. The conics tangent to $I_1L$ and $I_1L'$ at $L$ and $L'$, respectively, will determine an involution of rays at $A$, as at any point of the plane, made up of the tangents from $A$ to the respective conics. $g$ is a double ray of this involution, and $AL$ and $AL'$ are corresponding rays. For $g$ joins $A$ to the point of intersection of the fixed tangents to the range of conics, and $L$ and $L'$ constitute one of the degenerate conics of the range. But this is the involution previously determined at $A$. Likewise, at $B$ and $C$. Now, in this involution at $A$, $b$ and $c$ are corresponding rays; at $B$, $a$ and $c$ are corresponding rays. Hence, the particular conic of this range of those tangent to $I_1L$ at $L$ and $I_1L'$ at $L'$ which is itself tangent to $c$, is also tangent to $a$ and $b$. Since it also passes through $L$ and $L'$, it is one of the conics required. Each of the four self-corresponding points gives one such conic; viz., the conic tangent to the rays joining it to $L$ and $L'$ at $L$ and $L'$, and tangent to the three rays $a$, $b$, and $c$. In each case, $LL'$ is the polar of the self-corresponding point joined to $L$ and $L'$.

Hence, the following theorems, which are grouped for convenience.

**Theorem X.** *There are four conics tangent to three given rays and passing*

*through two given points. The ray joining the two given points being a common chord, the conics are determined as soon as its poles are known. Its four poles are the self-corresponding points of the involutory quadratic transformation of which the given lines constitute the fundamental triangle, and by which the given points are interchanged.*

These conics will all be real if the involutions determined at $A$, $B$, and $C$

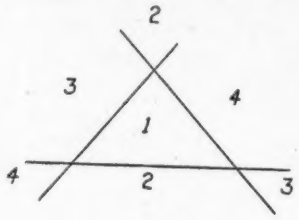


FIG. 5.

have double rays. For this to happen, the rays joining $A$, for instance, to $L$ and $L'$ must not be divided by $b$ and $c$. Hence, all the conics will be real if $L$ and $L'$ lie together in one of the four sections into which the given lines divide the plane[1] (Fig. 5). Otherwise, they are all imaginary, since two of the involutions have no double rays.

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## A SIMPLE METHOD OF CONSTRUCTING THE NORMALS TO A PARABOLA.

By S. G. BARTON, University of Pennsylvania.

We know by Harvey's Theorem that the feet of the three normals to a parabola from any point lie on a circle which passes through the vertex. The converse is also true, the normals whose feet lie on a circle through the vertex are concurrent. For let $(d, e)$ be the center of such a circle. Its equation then is $x^2 + y^2 - 2dx - 2ey = 0$, which intersects the parabola $y^2 = 4px$ in points whose ordinates are given by

$$y^3 - 8p(d - 2p)y - 32p^2 e = 0.$$

Since this lacks the second term the sum of its roots is zero which is the condition for concurrent normals.

The equation giving the ordinates of the feet of the normals through a point $(h, k)$ is

$$y^3 - 4p(h - 2p)y - 8p^2 k = 0.$$

But these equations must be equivalent, hence by equating coefficients $2d - 4p = h - 2p$ and $32e = 8k$; therefore $d = \frac{1}{2}h + p$ and $e = \frac{1}{4}k$.

[1] See Annie Dale Biddle, "Constructive Theory of the Unicursal Plane Quartic by Synthetic Methods," Univ. Calif. Publ. Math., Vol. 1, No. 2.

Hence to construct the normals from the point $(h, k)$, find the feet by constructing a circle whose center is $(\frac{1}{2}h + p, \frac{1}{4}k)$ and radius such that it passes through the vertex of the parabola. The intersections of the circle with the parabola other than the vertex are the feet of the three normals. If the point $(h, k)$ is on the convex side of the evolute of the parabola, the three normals are real and distinct; if on the evolute two are coincident, all three coinciding at the cusp; if on the concave side, two normals are imaginary. Under the same conditions the center of the circle will bear the same relation to the curve $(x - 2p)^3 = \frac{27}{2}py^2$.

If there is difficulty in locating the exact point of intersection, it will often be of aid to recall that the sum of the three ordinates must vanish.

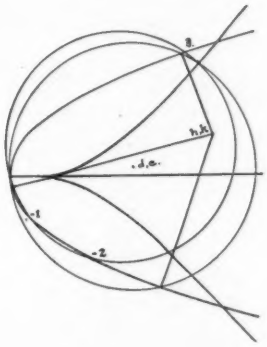
**A graphical solution for cubic equations** may be based upon the preceding construction for the normals of a parabola, since we have solved the cubic which gives the ordinates of the feet of the normals, namely,

$$y^3 - 8p(d - 2p)y - 32p^2 e = 0.$$

Hence in this manner we can solve any cubic. Let the cubic (which must be transformed so as to lack the second term), be of the form

$$y^3 - ay - b = 0.$$

Equating coefficients we get $d = a/8p + 2p$, $e = b/32p^2$. Hence construct any parabola and describe a circle whose center is $(d, e)$, as determined above, and radius such that it passes through the vertex. Then the ordinates of the inter-




sections, other than the vertex, are the roots of the equation. One parabola only need be drawn, but for convenience the unit of measure should be changed to suit the coefficients. It will usually be found desirable to choose $p = 0.2a$ approximately.

For example to solve the equation $x^3 - 7x - 3 = 0$, let $p$ be 0.5, whence $d = 2.75$ and $e = 0.75$.

This center and the intersections are shown on the figure. Whence we read the roots $-1$, $-2$ and 3. If the equation had been $x^3 - 700x - 6{,}000 = 0$, that is, the roots ten times as large, we would take $p = 5$ and the figure would be identical, the unit only being changed. The curve $(x - 2p)^3 = \frac{27}{2}py^2$, which gives the criterion for the nature of the roots is also shown. The roots are: one real and two imaginary, two equal or three real and unequal, according as the center $(d, e)$ is to the left of, on, or to the right of, this curve. The accuracy of the solution varies with the nature of the intersections.

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## SOME PROPERTIES OF THE NORMALS TO PARABOLAS.

By S. G. BARTON, University of Pennsylvania.

There are in general three normals to a parabola from any point. If the point lies on the evolute of the parabola, two of the normals coincide (all three coincide at the cusp); if on the concave side of the evolute two normals are imaginary; if on the convex side, the three normals are real and distinct. The following properties referring chiefly to normals drawn from points on the evolute were found in an investigation for another purpose.

Consider the parabola $y^2 = 4px$, whose evolute is $27py^2 = 4(x - 2p)^3$. If $(a, b)$ is a point on the evolute, then

(1) The equations of the coincident and single normals are respectively

$$y - b = \sqrt{\frac{a - 2p}{3p}}\,(x - a) \qquad \text{and} \qquad y - b = -2\sqrt{\frac{a - 2p}{3p}}\,(x - a)$$

and coördinates of the feet of these normals are

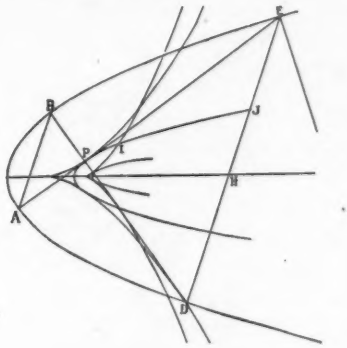
$$\left(\frac{a - 2p}{3},\ -2\sqrt{\frac{(a - 2p)p}{3}}\right) \qquad \text{and} \qquad \left(\tfrac{4}{3}(a - 2p),\ 4\sqrt{\frac{(a - 2p)p}{3}}\right).$$

Thus the abscissa of the foot of the single normal is four times that of the double normal, and the ordinate of the foot of the former is minus twice that of the latter. The abscissa of the foot of the double normal is one third of the difference between the abscissæ of the point $(a, b)$ and the cusp of the evolute, and the difference of the abscissæ of the feet is equal to the difference for the point and cusp.

(2) The $x$ intercepts are $\frac{1}{2}(a + 4p)$ and $\frac{1}{3}(4a - 2p)$. Hence the foot of the ordinate of the point $(a, b)$ trisects the distance between the intersections of the normals with the $x$ axis, being nearer the single normal. The difference of the two intercepts is $(a - 2p)$ which is the distance from the foot of the ordinate to the cusp. Thus the single normal cuts the $x$ axis four times as far from the cusp as the double normal. Similarly the foot of the abscissa trisects the distance between the intersections of the normals with the $y$ axis, being nearer the double normal. The $x$ intercept of the double normal is a mean proportional between the abscissæ of the points in which it intersects the parabola.

(3) The difference of the squares of the lengths of the normals (point to curve) equals three times the square of the difference of the abscissæ of the point and the cusp. The square on the line joining the feet of the normals is three times the square on the line joining the vertex with the foot of the double normal. The double normal (point to curve) is divided by the axis in a ratio which is twice that into which the axis divides the single normal. The locus of the intersection of two normals which are perpendicular to each other is the parabola $y^2 = p(x - 3p)$. This locus touches the evolute at $\left(\frac{1}{2}p, \pm \frac{\sqrt{2}}{2}p\right)$.

(4) These points and the normals through them are particularly interesting. The single normal through the upper of these points, $P$ in the figure, is $y = -\sqrt{2}(x - 4p)$. This line is normal to both parabolas, normal to the evolute at $P$ and tangent to the evolute at $D$.




If we draw the evolute of the parabola of perpendicular normals, $y^2 = p(x - 3p)$, which we call the second parabola, and its parabola of perpendicular normals, and so on we get a series of parabolas and evolutes each having the line $y = -\sqrt{2}(x - 4p)$ as a normal. The latus rectum of each parabola is one fourth that of the preceding and the cusp of each evolute is at the foot of the ordinate of the point in which the common normal cuts the parabola. $DB$ is the radius of curvature at $B$.

(5) The similar tangents and normals to the parabolas and evolutes at the points where the common normal cuts them are parallel. The tangent at the lower point of intersection in each case passes through the focus of the preceding parabola.

(6) The common normal and its correspondent constitute the envelope of the evolutes of the series of parabolas $y^2 = 4p'(x - 4p + 4p')$, in which $p'$ is the variable parameter. The parabolas considered are particular ones of the series.

(7) The length of the common normal is $6\sqrt{3}\,p$. It is trisected by the axis,

bisected at one point of intersection with the second parabola, and quadrisected by the other, smaller divisions being made by the other parabolas. The length of the double normal is $6\sqrt{6}\,p$. It is divided by the axis into parts which are as 1 to 5, and by the evolute into parts which are as 5 to 27 and 1 to 3.

(8) The line $AB$, joining the feet of the normals, passes through the focus (as is true for any pair of perpendicular normals), is $4\frac{1}{2}$ times $p$ in length, is trisected at the focus, is parallel to $DE$ (joining the other ends of the normals through $P$) and equal to $\frac{1}{3}DE$. Both $AB$ and $DE$ are bisected by the diameter through $P$ and are of course parallel to the tangent at its extremity.

(9) $DE$ is itself a normal to the parabola at $D$. If we draw the normal at $E$, then $DE$ bisects the angle between the other two normals through $E$, the two to the right forming with the axis an isosceles triangle.

(10) The cusp and the points $P$ and $I$ are collinear, $P$ being the point of bisection. The focus and the points $P$ and $J$ are collinear, $P$ being the point of quadrisection, and the join being perpendicular to the normal at $E$.

(11) $P$ is equidistant from $B$ and the focus. The vertex, $A$, $B$ and the middle point of $BD$ lie on a circle whose center is on $AE$. The point $I$ is equidistant from the cusp and $B$. The focus is equidistant from the intersections of the evolute with the parabola and $H$. A circle with $BD$ as a diameter passes through the vertex. $D$ and $B$ are equidistant from $J$ and $AB = JE$.

(12) The perpendicular bisector of $DH$ is tangent to the second parabola where $BD$ cuts it and the normal to the evolute at this point passes through $J$. The perpendicular to $DE$ at $H$ passes through $B$.

(13) The chord joining the vertex with the upper intersection of the parabola and evolute is perpendicular to $BD$ and trisected by it.

(14) The third normal from any point on the second parabola is trisected by the axis.

(15) The areas between the normal chords $AE$ and $BD$ and the parabola are equal. Moreover the three areas between the chords $BE$, $AB$, and $AD$ and the parabola are equal.

(16) From any point whatever $(a, b)$ the sum of the $X$ intercepts is $2(a + p)$, that is, twice the distance of the point from the directrix, and is constant for all points on a line perpendicular to the axis. Likewise the sum of the focal distances of the feet of the three normals is constant and equal to $(2a - p)$.

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## A SIMPLE ALGEBRAIC PARADOX.

By J. L. COOLIDGE, Harvard Universtiy.

Given two linear homogeneous complex equations

$$(1)\qquad \begin{aligned}(a + bi)(p + qi) + (c + di)(r + si) &= 0,\\ (a' + b'i)(p + qi) + (c' + d'i)(r + si) &= 0.\end{aligned}$$

In order that these should be compatible it is necessary and sufficient that

$$\begin{vmatrix} (a + bi) & (c + di) \\ (a' + b'i) & (c' + d'i) \end{vmatrix} = 0.$$

This complex equation is equivalent to the *two* real equations

$$(ac' - a'c) = (bd' - b'd),$$

$$(ad' + bc') = (a'd + b'c).$$

*Both* of these must be fulfilled if (1) is to subsist.

On the other hand equations (1) are equivalent to

$$\begin{aligned} ap - bq + cr - ds &= 0, \\ bp + aq + dr + cs &= 0, \\ a'p - b'q + c'r - d's &= 0, \\ b'p + a'q + d'r + c's &= 0. \end{aligned}$$

These are compatible when, and only when, a *single* equation is satisfied, namely

$$\begin{vmatrix} a & -b & c & -d \\ b & a & d & c \\ a' & -b' & c' & -d' \\ b' & a' & d' & c' \end{vmatrix} = 0.$$

Which answer is right?

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## ON A PURELY PROJECTIVE BASIS FOR THE THEORY OF INVOLUTION.

By D. N. LEHMER.

The theory of involution is usually based on foundations which are more or less metrical. It is customary to derive the more important properties by means of circles, or else by the use of the theory of the anharmonic ratio, which itself is usually based on metrical notions of areas and trigonometric functions. These methods have their advantages, but to the purist it seems unfortunate that the general, projective properties of involution should not be derived in a non-metrical way, and that the metrical properties should not be obtained, as is usual, by the introduction of the elements at infinity.

A purely projective discussion of the subject may be based on the following well-known a d easily derived theorem on the complete quadrangle:[1]

[1] See Reye, *Geometry of Position*, Holgate's translation, page 36.

*If two complete quadrangles, KLMN and K'L'M'N', are so related that the homologous sides KL and K'L' meet in a point A and the opposite pair, MN and M'N', in A'; LM and L'M' in B, and KN and K'N' in B'; LN and L'N' in C, while A, A', B, B' and C all lie on the same straight line u, then the sixth pair of sides, KM and K'M', will also meet on that line.*

The proof follows easily from Desargues's theorem and its converse. For consider the two triangles $KLN$ and $K'L'N'$ (Fig. 1): corresponding sides meet on $u$; therefore the lines $K'K$, $L'L$, and $N'N$ all meet in a point $S$. But by the triangles $LMN$ and $L'M'N'$ we observe that $MM'$ must also pass through $S$. Then since the lines joining corresponding vertices of the triangles $KMN$ and $K'M'N'$ are concurrent in $S$, we see that the pairs of corresponding sides meet in three collinear points $A'$, $B'$, $C'$ which must all lie on $u$.

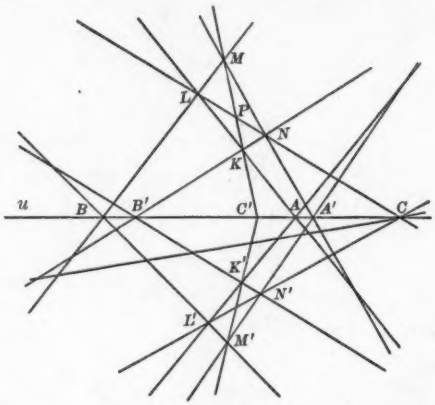


Fig. 1.

In this cut $A$, $A'$, and $B$, $B'$ are points on line $u$ and $KLMN$ and $K'L'M'N'$ are complete quadrangles constructed so as to have pairs of opposite sides passing through these points.

From the theorem it is seen that if $AA'$, $BB'$ and $C$ are given then $C'$ is uniquely determined, and that independently of the particular quadrangle employed to construct it. It is clear also that three pairs of points are not generally so placed on a line that the three pairs of opposite sides of a complete quadrangle may be drawn through them. As a definition, then:

*If three pairs of points on a line are such that the three pairs of opposite sides of a complete quadrangle may pass through them, the points are said to be three pairs in involution.* Any two pairs being given, as many other pairs may be constructed as desired, for choosing any point $C$, the other point that pairs with it, called its conjugate, may be found by the above linear construction. The solution, in detail, is as follows: Through $A$ and $A'$ draw any two lines. Cut them by any line through $C$ in the points $L$ and $N$. Join $L$ to $B$, and $N$ to $B'$, thus obtaining the points $M$ and $K$ on the lines through $A'$ and $A$ respectively. The line $MK$

determines $C'$. All the pairs $CC'$ thus obtained are said to belong to the same involution and taken together are said to separate the line in involution.

As $C$ traverses the line, $C'$ also traverses the line and the two point-rows thus obtained are projective. For fix $K$ and $N$, then $CN$ cuts out on $KA$ a point-row $L$, which projects from $B$ to a point-row $M$ on $NA'$, which, in turn, projects from $K$ to $C'$.

It follows from this last remark that $NC$ and $KC'$ furnish two projective pencils, and that their intersection, $P$, generates a conic. This conic meets the line $u$ in two points $F_1$ and $F_2$, real or imaginary, which are the *double points* in the involution; that is, they coincide with their conjugate points. Therefore: *An involution has two double points, or none.*

$B$ and $B'$ being a pair in the involution, the rays $NB$ and $KB'$ are corresponding rays. Therefore $NB$ is tangent to the conic at $N$, and $KB$ is tangent to it at $K$. $NK$ is then the polar of $B$ with respect to the conic. $B$ and $B'$ are thus conjugate with respect to the conic, and are thus harmonically separated by the two double points. We have then the theorem:

*Conjugate points in an involution are harmonic conjugates with respect to the double points.*

It is to be noted that, if the complete quadrangle is drawn with two pairs of opposite sides passing through the double points, the figure reduces to that used in defining four harmonic points.

**Metrical theory.** We introduce now that point at infinity the conjugate of which we will call the *center* and designate by $O$. Since conjugate points are harmonically separated by the double points it follows that $O$ is the middle point of $F_1F_2$.

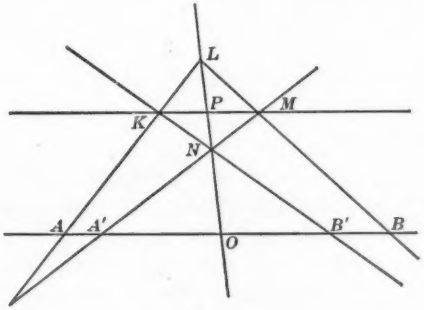


Fig. 2.

In constructing the center, the line $MK$ (Fig. 2) is drawn parallel to $u$. It is then seen that the triangle $MPL$ is similar to the triangle $OLB$, and the triangle $PNK$ to the triangle $ONB'$; whence we may write

$$\frac{OB}{MP} = \frac{OL}{PL}$$

and

$$\frac{OB'}{PK} = \frac{ON}{PN};$$

whence

$$\frac{OB \cdot OB'}{MP \cdot PK} = \frac{OL \cdot ON}{PL \cdot PN}. \tag{1}$$

Also from the similar triangles $OAL$ and $PKL$, combined with the similar triangles $ONA'$ and $MPN$, we have

$$\frac{OA \cdot OA'}{MP \cdot PK} = \frac{OL \cdot ON}{PL \cdot PN}. \tag{2}$$

By combining (1) and (2) we have the fundamental relation

$$OA \cdot OA' = OB \cdot OB',$$

or in words:

*The product of the distances from the center to a pair of conjugate points in the involution is constant.*

From this theorem all of the usual developments may easily be made. The corresponding theory of lines in involution may be obtained in the same way, or by applying the principal of duality.

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## BOOK REVIEWS.

EDITED BY W. H. BUSSEY.

*The Pell Equation.* By EDWARD EVERETT WHITFORD, Instructor in Mathematics in the College of the City of New York. Whitford, New York, 1912. iv + 193 pages. $1.00 postpaid.

Let $A$ be a given non-square positive integer and consider the equation

$$x^2 - Ay^2 = 1 \tag{1}$$

from the point of view of determining the positive integers $x$, $y$ which satisfy the equation. This is the so-called Pellian problem. Equation (1) is referred to as the Pell equation.

The principal purpose of the book under review is to give a history of the Pell equation (1) and of the more general equation $x^2 - Ay^2 = B$, where $B$ is a given integer (positive or negative). There is added a table of the simplest solution of (1) for each value of $A$ from 1,501 to 1,700 inclusive, similar tables having previously been given by other authors for the values of $A$ less than 1,501. The book contains also a bibliography of the Pell equation, with references to over 300 authors, and a table of continued fractions for $\sqrt{A}$.

Of the historical section, covering pages 1–101 of the text, we shall speak further.

The author begins with a discussion of Euler's error by which the name of Pell improperly came to be associated with equation (1). There follows an interesting discussion of the relation of the Pellian problem to that of approximating the square root of a non-square integer. An account is then given of the most ancient Hindu solutions (pp. 6–9), of the most ancient Greek solutions (pp. 9–13), of the work of Theon (pp. 13–15), Archimedes (pp. 15–21), Heron (pp. 21–22) and Diophantus (pp. 22–26). Emphasis is put upon the later work of the Hindus (pp. 26–39), and brief mention is made of that of the Arabs (pp. 39–41) and of the Europeans before the time of Fermat (pp. 41–46).

This history of the early and largely unsuccessful struggle of mathematicians with the difficulties of the Pellian problem is very instructive for those who are engaged now in the development and extension of science and knowledge.

The French mathematician Fermat (1601–1665) was the first to assert that equation (1), where $A$ is any non-square integer, always has an unlimited number of solutions in integers. To prove this and to obtain a method for finding all of these solutions he proposed it as a challenge problem to the English mathematicians. From his remarks about it, it is clear that he considered it a very difficult problem.

This challenge problem of Fermat's was solved by Lord Brouncker, at least so far as giving means for finding the solutions is concerned. On account of this success, the English mathematician Wallace congratulates his fellow-countryman Brouncker that he has "preserved untarnished the fame which Englishmen have won in former times with Frenchmen and has shown that England's champions in wisdom are just as strong as those in war." Fermat himself never published his solution of the problem.

It was Euler who first recognized the deep importance of the Pell equation for the general solution of the indeterminate equation of the second degree. He left several memoirs dealing with this question.

It was Lagrange who first proved in a rigorous manner that (1) always has solutions in integers. Of the work of Lagrange, Legendre said that it "must be considered the most important step which has been made up to the present time in the indeterminate analysis."

Our author treats further the work of Gauss and Dirichlet and of the relation of the Pellian problem to several other disciplines, namely, the following: quadratic forms, circle division, elliptic functions and hyperbolic functions.

The book on the whole will repay perusal both by the working mathematician and by the amateur in the theory of numbers.

R. D. Carmichael.

# PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

## PROBLEMS FOR SOLUTION.

**SPECIAL NOTICE.** In proposing problems and in preparing solutions, contributors will please follow the form established by the MONTHLY, as indicated on the following pages.

In particular, a solution should be preceded by the number of the problem, the name and address of the proposer, the statement of the problem, and the name and address of the solver.

The solution should then be given with careful attention to legibility, accuracy, brevity without obscurity, paragraphing and spacing, having in mind the form in which it will appear on the printed page.

Please use paper of letter size, write on one side only, leaving ample margins, put one solution only on a single sheet and include only such matter as is intended for publication.

Drawings must be made *clearly* and *accurately* and an extra copy furnished on a *separate sheet* ready for the engraver.

Unless these directions are observed by contributors, solutions must be entirely rewritten by the committee or else rejected.

Selections for this department are made two months in advance of publication.

Please send all solutions direct to the chairman of the committee.

MANAGING EDITOR.

### ALGEBRA.

**416. Proposed by H. O. HANSON, East Elmhurst, N. J.**

Find the $n$th term and the sum of $n$ terms of the series obeying the relation $u_i = u_{i-1} + 2u_{i-2}$ in terms of $n$ and the first two terms, $u_1$ and $u_2$, these two terms being arbitrary.

**417. Proposed by ELMER SCHUYLER, Brooklyn, New York.**

Solve $E(x^2) - E(3x) = 7$, where $E(m)$ is the largest integer in $m$. The value of $x$ is to be in the form $4 + y/32$ where $y$ is an integer.

### GEOMETRY.

**444. Proposed by S. A. COREY, Hiteman, Iowa.**

Let $ABCDE$ be a pentagon, plane or gauche, with sides $AB$, $BC$, $CD$, $DE$, and $EA$. Bisect $BC$ and $DE$ in $H$ and $K$ respectively. Extend $AB$ from $B$ to $B'$, and $AE$ from $E$ to $E'$. On $AB'$ take sects $AP$ and $AV$, and on $AE'$ take sects $AL$ and $AT$. Draw $AD$, $AC$, $AH$, $AK$, and $DT$. Let $a$, $b$, $c$, and $d$ equal $AL/AE$, $AT/AE$, $AV/AB$, and $AP/AB$, respectively. Extend (or contract) $AC$ from $C$ to $W$, and $AD$ from $D$ to $S$, making $AW = a \times AC$ and $AS = d \times AD$. Draw $LM$ and $PN$ parallel to, and of the same currency as, $AD$ and $AC$, respectively, and of lengths $c \times AD$ and $b \times AC$, respectively. Draw $AM$, $AN$, $ST$, and $WV$. Draw $DQ$ and $VX$ parallel to, and of the same currency as, $CB$ and $TS$ respectively. Prove that $2(ad + bc)(AK \times AH \times \cos KAH + KE \times HC \times \cos QDK) = AM \times AN \times \cos MAN + TS \times VW \times \cos WVX$.

*Suggestion*: Make use of the identity

$$[(w + x)a + (w - x)c][(z + y)d + (z - y)b] + [(w + x)b + (w - x)d][(z + y)c - (z - y)a] = 2(ad + bc)(wz + xy),$$

and solve with $w$, $x$, $y$, and $z$ vectors, in a manner similar to that employed by the proposer in the solution of problems 377 and 383 in the May and October, 1911, numbers of the MONTHLY. The results in certain special cases are instructive, *e. g.*, when one or more of the sides, $AC$, $CD$, $DE$, are zero; when two or more of the sides lie in the same right line; when all the sides are tangent to a sphere, circle, ellipse, etc.; when the angles given in the above equation are multiples of some angle, etc.

**445. Proposed by CLIFFORD N. MILLS, South Dakota State College.**

Given the perimeter of a right triangle $ABC$ and the perpendicular $BD$ falling from the right angle $B$ to the hypotenuse $AC$, to determine the sides of the triangle.

**446. Proposed by S. G. BARTON, University of Pennsylvania.**

Prove any one or more of the sixteen theorems, stated without proof, in the article in this issue of the MONTHLY (pages 182–184) on "Properties of the Normals to a Conic."

CALCULUS.

**366. Proposed by I. A. BARNETT, University of Chicago.**

Compute the definite integral $\int_a^b \sin^{-1} x dx$ where $0 \leqq a \leqq 1$ and $0 \leqq b \leqq 1$, by direct summation.

**367. Proposed by C. N. SCHMALL, New York City.**

Show that the volume enclosed by the surface $(x^2 + y^2 + z^2)^3 = (a^2x^2 + b^2y^2 + c^2z^2)^2$ is $\frac{4}{3}\pi(a^3 + b^3 + c^3)$.

MECHANICS.

**295. Proposed by B. F. FINKEL, Drury College.**

A homogeneous hollow cylinder, whose inner radius is half of its outer radius, rolls without slipping down a plane inclined at an angle $\alpha$ to the horizontal. Find its acceleration.

[From Prescott's *Mechanics of Particles and of Rigid Bodies.*]

NUMBER THEORY.

**218. Proposed by ELIJAH SWIFT, Princeton, N. J.**

If $p$ is prime and $> 3$, show that $\sum_{a=1}^{a=p-1} 1/a^2 \equiv 0 \pmod{p}$.

**219. Proposed by R. D. CARMICHAEL, Indiana University.**

Determine whether it is possible for a polygon to have the number of its diagonals equal to a perfect fourth power.

## SOLUTIONS OF PROBLEMS.

ALGEBRA.

A solution of 399 by WM. CULLUM and a solution of 400 by LOUIS O'SHAUGHNESSY were received too late for credit in the May issue.

**401. Proposed by R. D. CARMICHAEL, Indiana University.**

Prove the validity of Borda's series:

$$\log(x+2) = 2\log(x+1) - 2\log(x-1) + \log(x-2) + 2\left[\frac{2}{x^2-3x} + \frac{1}{3}\left(\frac{2}{x^2-3x}\right)^3 + \frac{1}{5}\left(\frac{2}{x^2-3x}\right)^5 + \cdots\right].$$

SOLUTION BY A. M. HARDING, University of Arkansas.

We have

$$\log(x+2) - 2\log(x+1) + 2\log(x-1) - \log(x-2)$$

$$= \log\frac{(x-1)^2(x+2)}{(x+1)^2(x-2)} = \log\frac{x^3-3x+2}{x^3-3x-2} = \log\frac{1+\dfrac{2}{x^2-3x}}{1-\dfrac{2}{x^2-3x}}$$

$$= 2\left[\frac{2}{x^2-3x} + \frac{1}{3}\left(\frac{2}{x^2-3x}\right)^3 + \frac{1}{5}\left(\frac{2}{x^2-3x}\right)^5 + \cdots\right],$$

which proves the result.

It should be observed that the series converges only for values of $x$ such that $x^3 - 3x > 2$. But $x^3 - 3x + 2 = (x+1)^2(x-2)$. Hence $x$ must be greater than 2.

Also solved by A. G. CARIS, F. M. MORGAN, HORACE OLSON, ELIJAH SWIFT, S. W. REAVES, and T. M. BLAKESLEE.

**402. Proposed by R. D. CARMICHAEL, Indiana University.**

Obtain other series similar to that of Borda given in the preceding problem (No. 401).

SOLUTION BY S. W. REAVES, University of Oklahoma.

If in the well-known series

$$(1)\quad \log(n+1) = \log n + 2\left[\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \cdots\right]$$

we set

$$n = \frac{(x-2k)(x+k)^2}{4k^3},$$

and therefore

$$n+1 = \frac{(x+2k)(x-k)^2}{4k^3}, \qquad \text{and} \qquad \frac{1}{2n+1} = \frac{2k^3}{x^3 - 3k^2x},$$

we shall have a generalized Borda's series

$$\log(x+2k) = 2\log(x+k) - 2\log(x-k) + \log(x-2k)$$
$$(2)\qquad + 2\left[\frac{2k^3}{x^3-3k^2x} + \frac{1}{3}\left(\frac{2k^3}{x^3-3k^2x}\right)^3 + \frac{1}{5}\left(\frac{2k^3}{x^3-3k^2x}\right)^5 + \cdots\right].$$

For $k = 1$, (2) becomes Borda's series.

Again, if in (1) we set

$$n = \frac{(x-3k)(x+k)^3}{16k^3x},$$

we shall have

$$\log(x+3k) = 3\log(x+k) - 3\log(x-k) + \log(x-3k)$$
$$(3)\qquad + 2\left[\frac{8k^3x}{x^4-6k^2x^2-3k^4} + \frac{1}{3}\left(\frac{8k^3x}{x^4-6k^2x^2-3k^4}\right)^3 + \cdots\right].$$

Also solved by F. M. MORGAN, T. M. BLAKESLEE, and A. M. HARDING.

## GEOMETRY.

**410. Proposed by A. H. HOLMES, Brunswick, Maine.**

Given a focus and two tangents to an ellipse, prove that the locus of the foot of the normal corresponding to either tangent is a straight line.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let $SP$ and $SP'$ be the fixed tangents, $F$ the fixed focus, $F'$ the other focus, and $N$ the foot of the normal corresponding to $SP$. Draw $FE$ and $F'E'$ perpendicular to $SP$, and $FG$ and $F'G'$ perpendicular to $SP'$. Draw $FF'$ cutting $SP$ in $L$.

In any ellipse the product of the perpendiculars from the foci on any tangent is constant, *i. e.*, $EF \times E'F' = GF \times G'F' = \text{constant}$.

Hence $E'F'/G'F' = GF/EF = \text{constant}$, and the locus of $F'$ is a fixed line $SF'$.

Now the lines $PS, PF, PN, PF'$ form an harmonic pencil, and $L, F, N, F'$ are harmonic points.

Hence the lines $SL, SF, SN, SF'$ form an harmonic pencil.

But $SL, SF, SF'$ are fixed. Therefore, $SN$ is a fixed line.

Likewise the foot of the normal corresponding to the other tangent $SP'$ lies on a fixed line which is the harmonic conjugate of $SP'$ with respect to $SF$ and $SF'$.

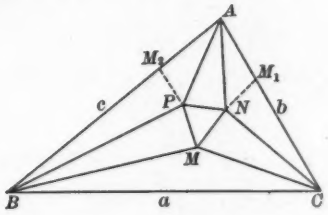
**431. Proposed by E. M. MORGAN, Dartmouth College.**

Trisect the angles of the triangle $ABC$ and let the trisectors nearest each side meet in the respective points $M, N, P$. Prove by trigonometry that the triangle $MNP$ is equilateral.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let $a, b, c$ denote the sides $BC, CA, AB$ respectively. Construct the points $M_1$ and $M_2$ on $CA$ and $AB$ respectively, such that $CM_1 = CM$ and $BM_2 = BM$.

Then $MN = M_1N$ and $MP = M_2P$.




If $3\alpha, 3\beta, 3\gamma$ represent the angles $A, B, C$ we have

$$\overline{M_1N}^2 = \overline{AN}^2 + \overline{AM_1}^2 - 2AN \times AM_1 \cos\alpha,$$

or

$$\overline{MN}^2 = \overline{AN}^2 + (b - CM)^2 - 2AN(b - CM)\cos\alpha.$$

Also

$$PN^2 = \overline{AN}^2 + \overline{AP}^2 - 2AN \times AP \cos\alpha.$$

Hence

$$\overline{MN}^2 - \overline{PN}^2 = (b - CM)^2 - \overline{AP}^2 - 2AN(b - CM)\cos\alpha + 2AN \times AP \cos\alpha,$$

$$= (b - CM - AP)(b - CM + AP - 2AN\cos\alpha).$$

Now

$$CM = \frac{\sin\beta}{\sin(\beta+\gamma)}a, \qquad AP = \frac{\sin\beta}{\sin(\alpha+\beta)}c, \qquad AN = \frac{\sin\gamma}{\sin(\alpha+\gamma)}b.$$

Also

$$b = \frac{\sin 3\beta}{\sin 3\alpha}a, \qquad c = \frac{\sin 3\gamma}{\sin 3\alpha}a.$$

Hence

$$b - 2AN\cos\alpha - CM + AP$$

$$= \left[1 - \frac{2\sin\gamma\cos\alpha}{\sin(\alpha+\gamma)}\right]b + \frac{\sin\beta}{\sin(\alpha+\beta)}c - \frac{\sin\beta}{\sin(\beta+\gamma)}a$$

$$= \frac{\sin(\alpha-\gamma)}{\sin(\alpha+\gamma)}b + \frac{\sin\beta}{\sin(\alpha+\beta)}c - \frac{\sin\beta}{\sin(\beta+\gamma)}a$$

$$= \left[\frac{\sin(\alpha-\gamma)\sin 3\beta}{\sin(\alpha+\gamma)\sin 3\alpha} + \frac{\sin\beta\sin 3\gamma}{\sin(\alpha+\beta)\sin 3\alpha} - \frac{\sin\beta}{\sin(\beta+\gamma)}\right]a$$

$$= \left[\frac{\sin(\alpha-\gamma)\sin 3\beta}{\sin(\alpha+\gamma)} + \frac{\sin\beta\sin 3\gamma}{\sin(\alpha+\beta)} - \frac{\sin\beta\sin 3\alpha}{\sin(\beta+\gamma)}\right]\frac{a}{\sin 3\alpha}$$

$$= \{\sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma)] + \sin\beta[3 - 4\sin^2(\alpha+\beta)]$$
$$- \sin\beta[3 - 4\sin^2(\beta+\gamma)]\}\, a/\sin 3\alpha$$

$$= \{\sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma)] - 4\sin\beta[\sin^2(\alpha+\beta)$$
$$- \sin^2(\beta+\gamma)]\}\, a/\sin 3\alpha$$

$$= \{\sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma)] - 2\sin\beta[\cos 2(\beta+\gamma)$$
$$- \cos 2(\alpha+\beta)]\}\, a/\sin 3\alpha$$

$$= \{\sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma)] - 4\sin\beta[\sin(\alpha-\gamma)\sin(\alpha+2\beta+\gamma)]\}$$
$$a/\sin 3\alpha$$

$$= \sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma) - 4\sin\beta\sin(\alpha+2\beta+\gamma)]\, a/\sin 3\alpha$$

$$= \sin(\alpha-\gamma)[1 + 2\cos 2(\alpha+\gamma) + 2\cos(\alpha+3\beta+\gamma)$$
$$- 2\cos(\alpha+\beta+\gamma)]\, a/\sin 3\alpha.$$

Now

$$\cos(\alpha+3\beta+\gamma) = -\cos 2(\alpha+\gamma)$$

and

$$\cos(\alpha+\beta+\gamma) = \cos 60^\circ = \tfrac{1}{2}.$$

Hence $b - 2AN\cos\alpha - CM + AP = 0$, and $MN = PN$.

Likewise $MP = PN$.

Also solved by T. M. BLAKESLEE.

## CALCULUS.

**335. Proposed by W. R. LEBOLD, Cambridge, Ohio.**

Let $\rho = F(\theta, \phi)$ be the equation in polar coordinates of a closed surface. Show that the volume of the solid bounded by the surface is equal to the double integral

$$\tfrac{1}{3}\int\int \rho\cos\gamma\, d\sigma$$

extended over the whole surface, where $d\sigma$ represents the element of area, and $\gamma$ the angle which the radius vector makes with the external normal. [Goursat-Hedrick, *Analysis*, p. 325, ex. 9.]

SOLUTION BY S. W. REAVES, University of Oklahoma.

We may take for element of volume $dV$ the cone (or pyramid) having the pole for vertex and the element of area $d\sigma$ for base. The altitude of this cone is clearly $\rho \cos \gamma$, and hence its volume is $dV = \frac{1}{3}\rho \cos \gamma d\sigma$. Hence the volume $V$, which is the sum of all the elements of volume, is, by the theory of definite integrals,

$$V = \tfrac{1}{3}\iint \rho \cos \gamma d\sigma,$$

where the integration is to be taken over the entire surface.

**344. Proposed by B. F. FINKEL, Drury College.**

Solve the differential equation,

$$\frac{d^2y}{dx^2}(a^2 - x^2 - y^2) = \left(x\frac{dy}{ds} - y\frac{dx}{ds}\right)\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}.$$

SOLUTION BY A. M. HARDING, University of Arkansas.

The equation may be written in the form

$$a^2 - x^2 - y^2 = \left(x\frac{dy}{ds} - y\frac{dx}{ds}\right)\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \left(x\frac{dy}{ds} - y\frac{dx}{ds}\right)\rho,$$

where $\rho$ is the radius of curvature at any point of the curve whose equation is sought.

Change to polar coördinates. Let $\phi$ be the angle the tangent makes with the initial line, and $\psi$ the angle between the radius vector and the tangent, *i. e.*, $\phi = \theta + \psi$. Then

$$\frac{dr}{d\theta} = r \cot \psi$$

and

$$\rho = \frac{\frac{ds}{d\theta}}{\frac{d\phi}{d\theta}} = \frac{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}{1 + \frac{d\psi}{d\theta}} = \frac{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}{1 + \frac{d\psi}{dr}\cdot\frac{dr}{d\theta}} = \frac{r \csc \psi}{1 + r \cot \psi \frac{d\psi}{dr}}.$$

Also

$$\frac{dy}{ds} = \sin \phi, \quad \text{and} \quad \frac{dx}{ds} = \cos \phi.$$

Substituting, we obtain

$$a^2 - r^2 = r(\cos\theta \sin\phi - \sin\theta \cos\phi) \cdot \frac{r \csc\psi}{1 + r \cot\psi \dfrac{d\psi}{dr}}$$

$$= r \sin\psi \frac{r \csc\psi}{1 + r \cot\psi \dfrac{d\psi}{dr}} = \frac{r^2}{1 + r \cot\psi \dfrac{d\psi}{dr}},$$

whence

$$\cot\psi d\psi = \frac{2r^2 - a^2}{r(a^2 - r^2)} dr$$

or

$$\log \sin\psi = \tfrac{1}{2} \log \frac{1}{r^2(a^2 - r^2)} + \tfrac{1}{2} \log c,$$

*i. e.*,

$$\sin^2\psi = \frac{c}{r^2(a^2 - r^2)}.$$

But

$$\sin^2\psi = \frac{r^2}{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}.$$

Therefore

$$\frac{c}{r^2(a^2 - r^2)} = \frac{r^2}{r^2 + \left(\dfrac{dr}{d\theta}\right)^2},$$

whence

$$\frac{\sqrt{c}dr}{\sqrt{-cr^2 + a^2r^4 - r^6}} = d\theta.$$

Let $r^2 = x$ and obtain

$$\frac{\frac{1}{2}\sqrt{c}dx}{x\sqrt{-c + a^2x - x^2}} = d\theta.$$

Integrate and obtain

$$\tfrac{1}{2} \sin^{-1} \frac{a^2r^2 - 2c}{r^2\sqrt{a^4 - 4c}} = \theta + \left(\frac{\pi}{4} - \beta\right),$$

after putting $r^2$ for $x$. Whence

$$\frac{a^2r^2 - 2c}{r^2\sqrt{a^4 - 4c}} = \cos(2\theta - 2\beta) = \cos 2\theta \cos 2\beta + \sin 2\theta \sin 2\beta.$$

Change to rectangular coördinates

$$[a^2 - \sqrt{a^4 - 4c} \cos 2\beta]x^2 - 2 \sin 2\beta \sqrt{a^4 - 4c}\, xy + [a^2 + \sqrt{a^4 - 4c} \cos 2\beta]y^2 = 2c.$$

Now introduce a new constant $k^4 = a^4 - 4c$. The equation may then be

written

$$[a^2 - k^2 \cos 2\beta]x^2 - 2k^2 \sin 2\beta \cdot xy + [a^2 + k^2 \cos 2\beta]y^2 = \frac{a^4 - k^4}{2}.$$

This is the equation of a system of conics, where the distance from the center to the focus is $k$, and where $\beta$ is the angle between the major axis and the $x$-axis.

Also solved by ELIJAH SWIFT, who obtained the result $\phi = \int \frac{dr}{r\sqrt{cr^3(r^3 - a) - 1}}$, and by C. N. SCHMALL and who obtained $xy = c$.

The above solution by Professor HARDING is correct.

## CALCULUS.

**345. Proposed by C. N. SCHMALL, New York City.**

Of all quadrilaterals which can be formed from four given sides, that which is inscriptible in a circle has the greatest area. [GOURSAT-HEDRICK, page 133, example 5.]

*Proposer's Remark.* Also show that when only three sides are known the length of the fourth side of the maximum quadrilateral is the root of a cubic equation.

### SOLUTION BY THE PROPOSER.

Let the area $= u = \Delta ABC + \Delta ADC$. That is,

$$u = \tfrac{1}{2}(ab \sin \phi + cd \sin \psi). \tag{1}$$

Also

$$\overline{AC}^2 = a^2 + b^2 - 2ab \cos \phi$$
$$= c^2 + d^2 - 2cd \cos \psi.$$

Hence

$$a^2 + b^2 - 2ab \cos \phi = c^2 + d^2 - 2cd \cos \psi. \tag{2}$$

Now for a maximum we have from (1)

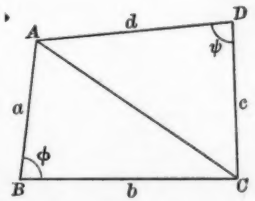
$$\frac{\partial u}{\partial \phi} = \frac{1}{2}\left( ab \cos \phi + cd \cos \psi \frac{\partial \psi}{\partial \phi} \right) = 0, \tag{3}$$

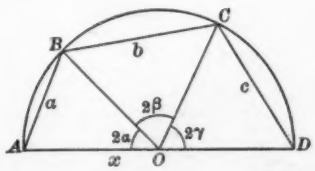
and from (2)

$$0 = 2ab \sin \phi - 2cd \sin \psi \frac{\partial \psi}{\partial \phi}. \tag{4}$$

Equating the values of $\partial\psi/\partial\phi$ obtained from (3) and (4), and reducing, we get

$$\frac{\cos \phi}{\sin \phi} = -\frac{\cos \psi}{\sin \psi}.$$



Hence $\tan \phi = -\tan \psi$, and $\phi + \psi = \pi$; that is, the opposite angles of the maximum quadrilateral are supplementary and hence the quadrilateral is cyclic.

Again, it is shown by elementary geometry, that when three sides of a maximum quadrilateral are given, the fourth side is the diameter of the circumscribing circle of the figure. Hence, to find this diameter $AD$, let $AD = x$, $AB = a$, $BC = b$, $CD = c$; and let the angles subtended by $a$, $b$, $c$, at the center be $2\alpha$, $2\beta$, $2\gamma$.

Then

$$\alpha + \beta + \gamma = \frac{\pi}{2} \tag{1}$$

and

$$\sin (\alpha + \beta + \gamma) = 1,$$

that is,

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma + 2 \sin \alpha \sin \beta \sin \gamma = 1. \tag{2}$$

But $\sin \alpha = a/x$; $\sin \beta = b/x$; $\sin \gamma = c/x$. Substituting these values in (2), we have

$$x^3 - x(a^2 + b^2 + c^2) - 2abc = 0. \tag{3}$$

This cubic has two negative roots and one positive root, as may be seen by putting $x = 0, -a, -\infty$. The positive root is the value of $x$ required. We can then describe a semicircle having this root as a diameter and place in it the chords $a$, $b$, $c$, in any order; for the equation (3) involves $a$, $b$, $c$, symmetrically.

*Note.*—The cubic (3) can also be derived as follows:

Draw the diagonals $AC$ and $BD$. Then

$$AC = \sqrt{x^2 - c^2}, \qquad BD = \sqrt{x^2 - a^2}.$$

Hence, by *Ptolemy's Theorem* we have

$$\sqrt{x^2 - c^2} \cdot \sqrt{x^2 - a^2} = ac + bx,$$

whence

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0,$$

as before.

Also solved by ELIJAH SWIFT, ELMER SCHUYLER, and A. M. HARDING.

## MECHANICS.

**273. Proposed by F. P. MATZ, Reading, Pa.**

A person is placed on a perfectly smooth surface. How may he get off.

SOLUTION BY S. W. REAVES, University of Oklahoma.

He should throw some object, for example his hat, in the opposite direction to that in which he wishes to go. The reaction, or "recoil," will cause him to slide to the edge of the smooth surface.

**284. Proposed by C. N. SCHMALL, New York, N. Y.**

A cylindrical vessel standing upright on a horizontal plane is kept constantly full of water by an automatic device. Determine at what height in its side a small orifice should be made,

so that the water may spout through it to the greatest horizontal distance on the plane. What difference in the result when the cylinder stands on a shelf of known height above the plane?

SOLUTION BY J. B. SMITH, Hampden-Sidney, Va.

Let $a$ be the height of the cylinder and $b$ the height of the orifice above the plane. Let $v_0$ be the velocity of the jet at the orifice. Then

$$v_0 = \sqrt{2g(a-b)}, \quad x_0 = v_0 t, \quad \text{and} \quad y = b - \tfrac{1}{2}gt^2.$$

Hence,

$$y = b - \frac{x^2}{4(a-b)}$$

is the equation of the trajectory, the origin being at the base of the cylinder and the $y$-axis being the vertical through the orifice.

For $y = 0$, we have $x^2 = 4b(a-b)$. As $x^2$ is a maximum when $x$ is, we have

$$\frac{d(x^2)}{db} = 4(a-b) - 4b = 0,$$

$$\therefore\ b = a/2.$$

Hence the range will be a maximum when the height of the orifice is half that of the cylinder.

If the cylinder stands on a shelf of height $h$ above the plane, we wish to find the maximum range on the line $y = -h$. In this case we have

$$x^2 = 4(b+h)(a-b).$$

It follows that the range is a maximum when $b = (a-h)/2$. If $h = a$, the orifice must be at the bottom of the cylinder; if $h > a$, $b$ would be negative, but as this is impossible, the greatest range for the given cylinder is obtained by making the orifice at the bottom.

Also solved by HORACE OLSON and B. LIBBY.

NUMBER THEORY.

**200. Proposed by R. D. CARMICHAEL, Indiana University.**

Find the general solution, in relatively prime integers, of the equation $x^2 + y^2 = z^4$.

SOLUTION BY ARTEMAS MARTIN, Washington, D. C.

Let $x = p^2 - q^2$, $y = 2pq$; then

$$x^2 + y^2 = (p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2 = z^4.$$

Let $p = r^2 - s^2$, $q = 2rs$; then

$$p^2 + q^2 = (r^2 - s^2)^2 + (2rs)^2 = (r^2 + s^2)^2;$$

therefore

$$x^2 + y^2 = (p^2 + q^2)^2 = (r^2 + s^2)^4 = z^4.$$

Retracing,

$$x = (r^2 - s^2)^2 - (2rs)^2, \qquad y = 4rs(r^2 - s^2), \qquad z = r^2 + s^2;$$

therefore

$$[(r^2 - s^2)^2 - (2rs)^2]^2 + [4rs(r^2 - s^2)]^2 = (r^2 + s^2)^4,$$

where $r$ and $s$ may be any numbers prime to each other, one odd and the other even.

Let $r = 2$, $s = 1$, and we have

$$7^2 + 24^2 = 625 = 25^2 = 5^4.$$

Let $r = 3$, $s = 2$; then we get

$$119^2 + 120^2 = 169^2 = 13^4.$$

Let $r = 4$, $s = 1$, and we find

$$161^2 + 240^2 = 289^2 = 17^4.$$

**203. Proposed by R. D. CARMICHAEL, Indiana University.**

Find solutions in integers of the equation

$$2x^2 + 1 = 3y^2. \tag{1}$$

I. SOLUTION BY E. E. WHITFORD, New York City.

Let $2x = z$. Then

$$z^2 - 6y^2 = -2. \tag{2}$$

By inspection the solution in smallest positive integers is $z_1 = 2$, $y_1 = 1$. The problem will not lose in generality if the solution be limited to positive integers. To represent concisely all the positive solutions without exception and without repetition I have derived the following formula:

$$z + y\sqrt{6} = \frac{(z_1 + y_1\sqrt{6})^{2k+1}}{2^k},$$

where

$$z_1 = 2, \qquad y_1 = 1, \qquad k = 0, 1, 2, 3, \cdots.$$

Therefore

$$z = 2, \quad 22, \quad 218, \quad 2158, \quad \cdots,$$

$$y = 1, \quad 9, \quad 89, \quad 881, \quad \cdots.$$

Since the values of $z$ are even each set gives a solution of equation (1).

$$x = 1, \quad 11, \quad 109, \quad 1079, \quad 10681, \quad \cdots,$$

$$y = 1, \quad 9, \quad 89, \quad 881, \quad 8721, \quad \cdots.$$

These results were obtained by Euler in his "Algebra," by de la Roche in his "Larismetique" (1520), who copied from the "Triparty" of Chuquet (1484); and probably by nearly everyone who tried, before decimal fractions came into common use, to find the approximate value of the square root of 6.

Equation (2) is a slightly generalized form of the Pell equation $x^2 - Ay^2 = 1$.

The Pell equation can be made an aid to the solution of all indeterminate equations of the second degree in two unknowns.*

II. Solution by Richard Morris, Rutgers College.

This equation is of the form $Dx^2 - Cy^2 = -H$ (see Chrystal's *Algebra*, Part II, page 450) and the integral values of $x$ and $y$ are dependent upon the expression of $\sqrt{CD}/D$ as a continued fraction, the value of $x$ being equal to the numerator and that of $y$ to the denominator of those convergents, where

$$(-1)^n M_n = -H.$$

Let $n$ denote the number of the convergent, $A_n$ the quotients, $p_n/q_n$ the terms of the convergent, and $M_n$ the $(n+1)$th rational divisor, and arrange these quantities in column form to exhibit the solutions. There will be an infinite number of solutions since $(-1)^n M_n = -1$ for all the odd convergents.

| $N$ | $A_n$ | $p_n$ | $q_n$ | $M_n$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 4 | 5 | 4 | 2 |
| 3 | 2 | 11 | 9 | 1 |
| 4 | 4 | 49 | 40 | 2 |
| 5 | 2 | 109 | 89 | 1 |
| 6 | 4 | 485 | 396 | 2 |
| 7 | 2 | 1,079 | 881 | 1 |
| 8 | 4 | 4,801 | 3,920 | 2 |
| 9 | 2 | 10,681 | 8,721 | 1 |

Thus $x = 1$, $y = 1$; $x = 11$, $y = 9$; $x = 109$, $y = 89$, etc., are integral solutions.

Also solved by C. E. Githens.

## MISCELLANEOUS QUESTIONS.

Edited by R. D. Carmichael.

*Note.*—This department is designed to furnish a forum for the discussion of live questions in the teaching of collegiate mathematics and of difficulties actually encountered by any one who may desire the reaction of other workers in the same field. It is believed that much benefit may accrue from the interchange of ideas along these lines, and it is hoped that a widespread interest will develop in maintaining the department.

Editors.

### QUESTION.

**12.** In view of the notation used by Professor Slobin in his "Note on Certain Algebraic Equations" in the April issue, pages 113–115, a discussion would be desirable as to the best notation for complex roots in general, and in particular for eliminating the conspicuous ambiguities introduced by the notation above cited.

### REPLIES.

**7.** What place should be given to the history of mathematics in courses for prospective high school teachers, and why?

* For an account of such solutions see "The Pell Equation," E. E. Whitford, *Publications of the College of the City of New York*, 1912, pages 60, 64, ff.

## I. Remarks by G. A. Miller, University of Illinois.

The history of mathematics is one of the most useful phases of applied mathematics since it enables us to measure an important type of intellectual development at various periods. A study of some of the applications of mathematics frequently makes particular subjects both easier and more interesting. It also establishes contact with other subjects and thus tends towards greater unity in the student's knowledge.

It seems to me that the prospective high school teacher should have a course in the history of mathematics extending over at least five hours per week for one semester. One difficulty is to find teachers in our universities and normal schools who are really prepared and willing to give such a course. After various trials I have come to the conclusion that a course in the history of elementary mathematics is very difficult and requires an unusual amount of preparation on the part of the teacher.

In the opening article of volume 14, February, 1914, of the *Bibliotheca Mathematica*, the noted mathematical historian, G. Eneström, considers the question of teaching the history of mathematics in the secondary schools. He emphasizes the fact that it is very difficult to secure accurate information as regards many of the most important historical questions in view of the many incorrect statements which appear even in the best histories of mathematics. Those who read only the English language are at a great disadvantage since no extensive history of mathematics has ever appeared in this language.

Notwithstanding these difficulties it seems to me that a course in the history of mathematics should be regarded as an essential element in the training of the high school teacher. Such a course should be one of the most useful means to impress the important fact that statements found in books must generally be subjected to a careful thought analysis in order to render them helpful for arriving at truths. It should also give a broader meaning to the various mathematical subjects and lead to a deeper interest by adding the dynamic element of development to the various mathematical concepts.

## II. Remarks by W. T. Stratton, Kansas State Agricultural College.

Every prospective high school teacher should be given an opportunity to become familiar with the history of the mathematics he will teach. The course need not be very extensive but should have at least two or three hours per week throughout the senior year devoted to it. This course should not only take up the history of the different subjects but should also go into the biographies of some of the best known mathematicians. Mathematics has a history that begins with the history of the race. It has held the attention of some of the greatest men of all times. The prospective mathematics teacher should be given a chance to receive the inspiration that will come to him by a study of the lives of these men. As he goes out to teach in the high schools he will find that new spirit and new life can often be put into his courses if he is able to point out

occasionally some interesting historical facts connected with the development of the subject in hand. Pupils, too, are always interested in men and if the subject is developed in connection with the facts concerning men who have made mathematical history they will likely look upon the subject with more interest and enthusiasm.

A course in the history of mathematics will enlarge the teacher's vision. He should be able to see the points of difficulty in the development of the subject by the race and should be better prepared to take the pupils over these difficulties. He will see the dependence and interdependence of arithmetic, algebra, and geometry. The three merge into each other so gradually that the dividing line cannot be determined. Nothing will help more than a good course in the history of the subject to give the teacher the proper attitude toward these divisions.

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## NOTES AND NEWS.

UNDER THE DIRECTION OF FLORIAN CAJORI.

As previously announced in these columns, Professor Cajori will be in Europe during the coming year. During his absence, Professor W. D. Cairns, of Oberlin College, Ohio, will act as chairman of the committee on Notes and News, and all contributions of this character should be sent directly to him.

Dr. C. L. BOUTON of Harvard University has been promoted to an associate professorship of mathematics.

The *Educational Review* for May contains an article on "Mathematics for culture" by Professor N. J. LENNES, of the University of Montana.

Dr. C. KÜSCHKE, formerly instructor at the University of California, where he took his doctorate, has accepted an instructorship in mathematics for the coming year at the University of Washington. He has recently been attending lectures at the University of Chicago.

Dr. G. N. GREEN, of the College of the City of New York, has accepted an instructorship in mathematics at Harvard University.

Mr. LESTER HILL, formerly a graduate student at Columbia University and later at the University of Chicago, has been made assistant professor in mathematics at the University of Montana.

Professor N. J. LENNES, head of the department of mathematics at the University of Montana, will spend the coming summer abroad.

Bulletin No. 323, of the University of Texas, contains an article by Dr. EDWARD L. DODD on "The error-risk of the median compared with that of the arithmetic mean."

Mr. EDWARD KIRCHER, graduate student at the University of Illinois, has been appointed to an instructorship in mathematics at the Massachusetts Institute of Technology.

Professor C. RUNGE, of the University of Göttingen, was elected president for 1914 of the German Mathematical Society. According to the most recent list this society has 769 members.

Henry Holt & Company announce that the following two mathematical books are in press: "Analytic geometry of space," by VIRGIL SNYDER and C. H. SISAM; and "Plane analytical geometry," by L. W. DOWLING and F. E. TURNEAURE.

John Wiley and Sons have just published "The theory of numbers," by ROBERT D. CARMICHAEL as No. 13 of the series of mathematical monographs edited by Mansfield Merriman and Robert S. Woodward. Professor Carmichael is also author of No. 12 in this series entitled "The theory of relativity."

The Macmillan Company announces the publication of a new edition of Professor F. R. Moulton's *Introduction to Celestial Mechanics*. This has been revised and largely rewritten, but still preserves the general plan of the earlier edition.

Professor GEORGE N. BAUER has been appointed chairman of the department of mathematics of the college of science, literature, and the arts at the University of Minnesota for the year 1914–15.

An interesting article entitled "Elementary theorems on the hexahedron in non-euclidean geometry," by Dr. ARTHUR RANUM, of Cornell University, was published in the *Quarterly Journal of Pure and Applied Mathematics*, No. 178, 1914.

The annual dinner for instructors and fellows in mathematics, at the University of Chicago, given by the faculty of the department, occurred on the evening of May 15, 1914. The guest of honor was Dr. T. M. Putnam, of the University of California, who had just been promoted to the rank of associate professor and was on his way to Europe where he will spend the next half year on sabbatical vacation.

Professor F. CAJORI's History of Elementary Mathematics is being translated into the Japanese language by Mr. Seiich Sunagawa, of the Ehimeken Female Normal College.

The Macmillan Company has just published a book which will be of interest to the readers of the MONTHLY. It is entitled "Dialogues concerning two new sciences," by Galileo Galilei, translated from the Italian and Latin into English by HENRY CREW and ALFONSE DE SALVIO. This will be reviewed in a later issue of the MONTHLY.

A new part of the German edition of the large mathematical encyclopedia was recently published. It bears the title "Ansätze und allgemeine Methoden der Systemmechanik." The French edition recently published a part entitled: "Fonctions sphériques; generalizations diverses des fonctions sphériques."

An error was made in the March issue of the MONTHLY, page 86, in the review of A. W. STAMPER'S textbook on the Teaching of Arithmetic, where it was stated that the Austrian method of division is not given. A brief reference with explanation and incompleted division is found on pages 95–96 of this text.

The American Association for the Advancement of Science has appointed a Committee of One Hundred on scientific research, under the government, in the universities and in other institutions. Pure mathematics is represented on this committee by Professor G. A. MILLER, University of Illinois, and Professor E. H. MOORE, University of Chicago.

Special efforts are being made to secure as large a collection as possible of letters by and to LEONHARD EULER for publication in his complete works. In view of the fact that this publication is much more expensive than was expected, the committee in charge is anxious to secure more subscribers for these works, as well as more members of the Euler Society. The members of this society promise to contribute at least ten francs per year towards the expense of publishing Euler's complete works.

An "Elementary Theory of Equations" by Professor L. E. DICKSON, of the University of Chicago, has just been published by John Wiley and Sons. It contains a chapter of 13 pages on complex numbers, one of 33 pages on determinants, and one of 17 pages on resultants and discriminants. The whole book consists of 183 pages. A review of it will appear in a future number of the MONTHLY.

The next Summer meeting of the American Mathematical Society will be held at Providence, R. I., September 8, 9, 1914, in connection with the celebration of the one hundred and fiftieth anniversary of Brown University.

The fourth instalment of the National Geometry Syllabus, which is being reprinted in the *Irish Journal of Education*, appeared in the May issue. It begins with Section B, Logical Considerations.

The summer quarter of the University of Chicago opened June 15th with the usual large attendance. The registration in mathematics is especially strong in the advanced courses.

Dr. C. E. GITHENS, who is principal of the Union School at Wheeling, West Va., was recently elected superintendent of the public schools of that city. He has long been a contributor to the problem department of the MONTHLY.

Mr. J. D. ESHLEMAN has been elected to an instructorship in mathematics at Western Reserve University, and Mr. L. E. WILLIAMS to a similar position in the Georgia School of Technology. Both have been graduate students of the University of Chicago during the past year.

Professor L. E. DICKSON, of the University of Chicago, will spend the first semester, Autumn 1914, in residence at the University of California lecturing along the lines of his special work in number theory. He was at one time a member of the faculty of the University of California.

Miss MARY E. WELLS, formerly instructor at Mount Holyoke College and now a graduate student at the University of Chicago, has been appointed instructor in mathematics at Oberlin College for the coming year, during which time Dr. MARY E. SINCLAIR will be on leave of absence. A recent announcement that Dr. T. H. GRONWALL, of Princeton University, had been appointed to an assistant professorship in mathematics at Oberlin College was erroneous. His appointment was at Princeton University.

Teachers of mathematics who are interested in the wide scope of the applications of mathematics will find it interesting to read a paper by Professor I. J. SCHWATT, of the University of Pennsylvania, entitled "The Applications of the Calculus to the Medical Sciences," published in the *American Journal of the Medical Sciences*, March, 1914, page 409. Since this journal does not usually contain material of a mathematical nature, a special mention here seems desirable.

The firm of Allyn and Bacon has published a trigonometry by Professors E. J. WILCZYNSKI and H. E. SLAUGHT, of the University of Chicago. Among the interesting features of this text are the numerous well-selected applications, and the arrangement of the text in two distinct parts, the first containing only such theoretical development as is essential to the solution of triangles. A review of this book will appear in a later issue of the MONTHLY.

During the recent meeting of the Michigan Schoolmasters' Club at Ann Arbor an exhibit of early mathematical text-books from the university collections and the private libraries of Professors BEMAN, ZIWET and KARPINSKI was displayed in the library. A large number of photographs of mathematical manuscripts in European libraries and from the collection of Mr. G. A. Plimpton were loaned for the exhibit by Professor L. C. KARPINSKI.

The Council of Mathematical Teachers in New England has recently appointed a special committee on the status and welfare of mathematics in secondary schools, to investigate and report on current criticisms of high school mathematics. The membership of the committee is as follows: Mr. G. W. EVANS, Charlestown High School; Professor F. C. FERRY, Williams College; Mr. A. V. GALBRAITH, Middlesex School; Mr. F. P. MORSE, Revere High School; Mr. C. D. MESERVE, Newton High School; Professor S. E. SMITH, Mount Holyoke College; Miss H. R. PIERCE, Worcester High School; and Professor H. W. TYLER, Massachusetts Institute of Technology, *chairman*. Correspondence with persons having special information is invited.

The application of the new school law in Ohio will probably necessitate that those preparing to teach secondary school subjects and looking to certification by the state department of education, shall devote one fourth of their college work directly to preparation for teaching; fifteen semester hours to be of distinctly educational work including practice and observation courses, teachers training courses in secondary school subjects, psychology, history and principles of education, educational methods, etc., fifteen hours to be chosen from the fields of psychology, sociology, logic, ethics, philosophy, and education. Several of the Ohio colleges, among these Ohio State University, Oberlin College and Miami University, are enabling teachers to meet these new requirements at once by offering suitable courses in the summer sessions of 1914.

The annual meeting of the Society for the Promotion of Engineering Education was held at Princeton University, June 23–26, 1914. The papers were printed in advance in the Bulletin of the Society, so that only short abstracts were read at the meetings, thus leaving ample time for discussion. The prospectus stated that "It is necessary to have papers and discussions at a convention, but these are not more than half of the meeting. It would be worth while to come to Princeton, even if there were no papers, just to meet the one hundred and ninety-nine other teachers." At all events, it seems to be a decided step forward to print papers in advance and to spend less time in *reading* and more in *discussion* and in general intercourse.

At the mathematical conference held on April 3d at Ann Arbor, Professor THEODORE R. RUNNING, of the University of Michigan, presented a paper on "Graphical methods applied to the solution of algebraic equations," Miss KATHERINE G. HINE, of the Detroit Central High School, offered "Some class room suggestions," Mr. G. C. BARTOO, of the Jackson High School, discussed "The problem of reducing the number of failures in algebra," and Professor W. B. FORD, of the University of Michigan, talked on "The future of geometry." Next year the conference is planning to have a luncheon and informal discussion to replace part of the formal program. Professor L. C. KARPINSKI was elected chairman of the conference for 1914–1915 and Mr. EDWARD T. GEE, of the Detroit High School, secretary.

The editors of the MONTHLY acknowledge the receipt of the following journals on our exchange list: *School Science and Mathematics, The Mathematics Teacher, The Annals of Mathematics, The American Journal of Mathematics, Transactions of the American Mathematical Society, Bulletin of the Society for the Promotion of Engineering Education, Proceedings of the American Philosophical Society, The Monist, Popular Astronomy, Journal of the Royal Astronomical Society of Canada, The Mathematical Gazette, L'Intermédiaire des Mathématiciens, L'Enseignement Mathématique, Revue Semestrielle des Publications Mathématiques, Periodico di Matematica per L'Insegnamento Secondario, Bolletino di Biblografia e Storia delle Scienze Matematiche, Rendiconti del Circolo Matematico di Palermo, Giornale de*

*Matematiche di Battaglini, Nieuw Archief voor Wiskunde, Wiskundige Opgaven met de Oplossingen, Revista de la Sociedad Matemática Española, Iberica- el Progreso de las Ciencias y de sus Aplicaciones, The Tôhoku Mathematical Journal, The Journal of the Indian Mathematical Society.* It is the intention hereafter to mention more frequently in these notes such articles in our exchanges as are deemed to be of special interest to readers of the MONTHLY.

A mathematical colloquium will be held at Edinburgh, July 28–31, immediately following the Napier tercentenary celebration. There will be two lectures by Professor M. D'Ocagne, of Paris, on "Nomography"; four lectures by Professor H. W. Richmond, of Cambridge University, on "Infinity in geometry"; four lectures by Professor E. Cunningham, of Cambridge University, on "Critical studies of modern electrical theories"; and two lectures by Professor E. T. Whittaker, of the University of Edinburgh, on "The solution of algebraic and transcendental equations in the mathematical laboratory." The methods described by Professor Whittaker will be chiefly arithmetical, and the lectures will therefore be supplementary to those of Professor D'Ocagne on nomographic methods, which for most purposes are now said to be generally recognized as superior to the older graphical methods of calculation. The announcement states that "the establishment of nomography as a regular constituent of British mathematical teaching is much to be desired."

Readers of the MONTHLY are asked to give attention to the announcements of the various publishers of mathematical books found in this and other issues. These publishers are not only serving our readers by giving up-to-date information concerning new books on the historical and philosophical aspects of mathematics, as well as text-books in the various fields, but they are also contributing in a substantial manner toward the support of this journal.

Complete sets of the MONTHLY for the year 1913, being the first volume under the reorganization, are in constant demand, but the supply of certain numbers in that volume has been exhausted. These numbers are January and June, 1913. Since the last issue several extra copies of the January number have been sent in and now a few complete sets of volume xx are again available, and several other sets are complete except for the June number. Any one who can contribute an extra copy of either the January or the June number will confer a great favor upon the editors.

The next issue of the MONTHLY will appear in September. There are no issues in July and August. During the summer, Professor Finkel, the treasurer, will be at Boulder, Colorado, in charge of mathematics in the summer session of the University of Colorado. Business communications may still be addressed to him at Springfield, Missouri, where they will receive prompt attention. The Managing Editor will be in residence at the University of Chicago and may be addressed as usual at 5548 Kenwood Ave., Chicago, Ill.

7

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# THE APPARENT SIZE OF A CUBE.

By A. M. HARDING, University of Arkansas.

1. The apparent size of any surface is defined as the solid angle of a cone whose vertex is at the point of sight and whose directrix is the curve bounding the surface. The solid angle, being measured by the area cut by the cone from a unit sphere whose center is at the point of sight, may vary from zero to $4\pi$, the apparent size of the whole "sky." The apparent size of a surface as seen from a certain point of space is evidently a function of the position of the point of sight. Its value is given by the integral

$$\omega = \int \frac{\cos(n, r)}{r^2} dS \qquad (1)$$

extended over the whole surface, where $r$ is the radius vector from the point of sight to any point of the surface, $dS$ any element of surface, and $(n, r)$ the angle between $r$ and the positive direction of the normal to the surface.

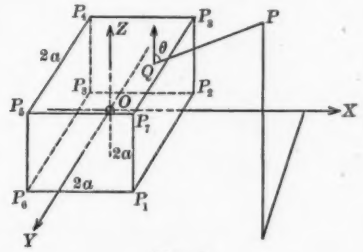


FIG. 1.

We shall take the center of the cube as origin and let the coördinate axes be parallel to the three edges of the cube. Let us consider first the solid angle subtended by the upper face of the cube, i. e., by the square $P_7P_3P_4P_5$. Let